Light diffraction by ultrasonic pulses: Analytical and numerical solutions of the extended Raman–Nath equations

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Light diffraction by ultrasonic pulses at normal incidence is studied as well in the Raman–Nath region (low frequencies) as in the Bragg region (high frequencies). Based on a generating function method, an analytical expression for the Raman–Nath-like diffraction has been derived and compared with earlier work. Only small corrections within its validity region were observed. In order to extend these existing theories toward higher frequencies, numerical expressions for the intensity of the diffracted light waves are obtained by means of the Laplace transform theory. This powerful method leads to the same results for Raman–Nath-like diffraction and can easily be applied for much higher frequencies, too. Examples are provided showing the frequency dependence of the ultrasound, the influence of the Raman–Nath parameter v, and the spectral composition of the pulse on the diffraction pattern. When the pulse approaches a continuous wave, both theories converge to known results. A general condition concerning the symmetry properties of the diffraction spectrum has been derived.

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INTRODUCTION

Light diffraction by ultrasound has been studied in the past by several authors. Since Raman and Nath[1] explained the first observations of this interaction made by Debye and Sears[2] in Washington and Lucas and Biquard[3] in Paris, by considering the ultrasonic beam as a moving phase grating, several more complete treatments were proposed.[4–9] These refinements cover a lot of interesting topics such as optical probing of superposed, adjacent, or profiled ultrasonic waves. However, not only the question how light is modulated in amplitude and frequency after interaction with ultrasound was investigated, also the inverse problems that deduce the disturbing ultrasonic field from a known diffraction pattern, were examined with considerable interest.[10,11] In nondestructive testing, the diffraction pattern is used to detect small defaults.[12]

As, in general, pulsed ultrasound is used instead of continuous plane waves for medical diagnosis or NDT, we generalized the existing theory and developed a new one for pulses. Starting from the Maxwell equations, Leroy[13] obtained an extended Raman–Nath system of differential equations for the amplitudes of the different orders arising when plane light waves are diffracted by N superposed ultrasonic waves. Knowing that pulses can be considered as a superposition of plane waves, it is our current interest to solve this system for arbitrary N.

We first developed a generalization of the extreme Raman–Nath-like diffraction theory for plane waves. The generating function method provides an analytic expression, being the second-order approximation of a series expansion in a parameter ρ_p, which is proportional to the squared ratio of the fundamental frequency of the ultrasonic pulse and the light frequency. As a special case, we find the expressions experimentally verified by Neighbors and Mayer.[14] The limits of validity of both methods are discussed and graphically illustrated.

A second generalization is based on the Laplace transform method.[15,16] Applying this powerful method, expressions for the diffracted light intensities can only be given in a numerical way, but, on the other hand, the range of validity becomes much larger. Our model provides reasonable results for any range of frequencies. Influences of both parameters ρ_p and v and the transition from continuous plane wave to pulses are demonstrated.

At last, a general theory concerning the symmetry properties of the diffraction pattern of light obtained after interaction with ultrasonic pulses at normal incidence is given.

I. GENERAL SYSTEM OF DIFFERENTIAL EQUATIONS DESCRIBING THE AMPLITUDES OF THE DIFFRACTED LIGHT BEAMS

Consider a plane-wave laser beam with wavelength λ and frequency ν, normally incident on a pulsed ultrasonic wave with repetition frequency ν_p^* and center modulation frequency ν_0^* propagating in a vessel filled with water. Choosing the geometry such that the sound beam, having width L, is propagating in the x direction and the light beam in the z direction, the time history of the refractive index of the medium disturbed by a pulse train can be expressed by the Fourier sine expansion:

$$\mu(x,t) = \mu_0 + \mu \sum_{j=1}^{\infty} \alpha_j \sin\left[2\pi j\left(\nu_p^* t - \frac{x}{\lambda_p^*}\right) + \delta_j\right], \qquad (1)$$

where ν_p^* is the repetition frequency, λ_p^* is the length of the pulse in the medium, μ_0 is the refractive index of the undisturbed medium, μ is the maximum variation of the refractive index, and $\mu \cdot \alpha_j$ and δ_j are the amplitude and phase of the jth Fourier component of the pulse (see Fig. 1).

In an earlier work,[13] Leroy derived that the amplitudes

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of the diffraction orders in the far field satisfy the extended Raman–Nath equations (2):

$$2\frac{d\Psi_n}{d\zeta}(\zeta)-\sum_{j=1}^{\infty}\alpha_j[\Psi_{n-j}(\zeta)\exp(-i\delta_j)$$
$$-\Psi_{n+j}(\zeta)\exp(i\delta_j)]=i\rho_p n^2\Psi_n(\zeta), \tag{2a}$$

with boundary conditions

$$\Psi_n(\zeta=0)=\delta_{n,0}\quad(n:-\infty\cdots+\infty), \tag{2b}$$

where

$$\zeta=(2\pi/\lambda)\mu z \tag{2c}$$

and

$$\rho_p=\lambda^2/\mu_0\mu\lambda_p^{*2}. \tag{2d}$$

Evaluated in the Raman–Nath parameter v, defined as $(2\pi/\lambda)\mu L$, with L the width of the sound beam, the solution $\Psi_n(v)$ is the amplitude of the diffracted light beam whose direction with regard to the incident laser beam is defined by the angle

$$\theta_n=-\arcsin(n\lambda/\lambda_p^*), \tag{3}$$

while its frequency will be $\nu-n\nu_p^*$ and the intensity is given by

$$I_{-n}(v)=\Psi_n(v)\cdot\Psi_n^*(v). \tag{4}$$

We now propose two methods to integrate this infinite system of coupled differential equations (2).

II. CALCULATION OF THE FAR-FIELD AMPLITUDES OF THE DIFFRACTION PATTERN BY MEANS OF A GENERATING FUNCTION METHOD

A. Transformation of the system of differential equations into one partial differential equation for the generating function

In their study of the diffraction of light by one single fundamental tone, Kuliasko *et al.*[17] successfully introduced the generating function method in order to integrate the simplified system of differential equations. We will use the same method to find the solution of (2) describing the diffracted intensities.

Let us consider $\Psi_n(\zeta)$ to be the coefficients of the Laurent expansion of an unknown generating function $G(\zeta,\eta)$, which we suppose to be holomorphic in an annular region with center O in the complex η plane.

So, we have

$$G(\zeta,\eta)=\sum_{n=-\infty}^{+\infty}\Psi_n(\zeta)\eta^n. \tag{5}$$

The coefficients $\Psi_n(\zeta)$ are then given by

$$\Psi(\zeta)=\frac{1}{2\pi i}\oint_C\frac{G(\zeta,\eta)}{\eta^{n+1}}d\eta, \tag{6}$$

whereby the contour C is any closed path within the annular region encircling the origin O once in a counterclockwise

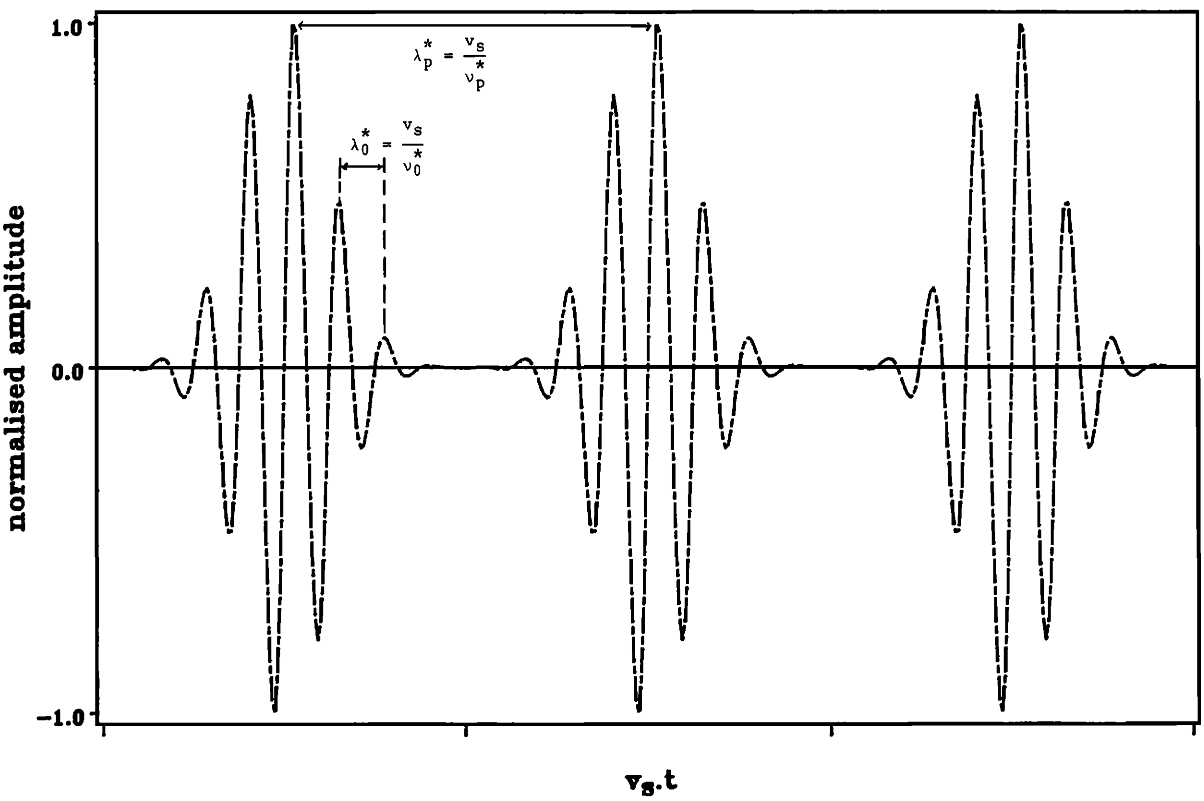

FIG. 1. Visualization of repetition (ν_p^*) and center modulation frequency (ν_0^*) of a typical pulsed ultrasonic sequence (v_s is the velocity of the sound in the medium).

sense. If we multiply both sides of (2a) by η^n and sum over all n, taking into account the expressions for the first- and second-order partial derivatives of $G(\zeta,\eta)$ with respect to η, we find that

$$2\frac{\partial G}{\partial \zeta}(\zeta,\eta) - \sum_{j=1}^{+\infty}\alpha_j[\eta^j\exp(-i\delta_j)-\eta^{-j}\exp(i\delta_j)]G(\zeta,\eta) = i\rho_p\left(\eta^2\frac{\partial^2 G}{\partial\eta^2}(\zeta,\eta)+\eta\frac{\partial G}{\partial\eta}(\zeta,\eta)\right). \tag{7a}$$

The boundary conditions (2b) and the definition of the function $G(\zeta,\eta)$ lead to the following condition with respect to G:

$$G(\zeta=0,\eta)=1. \tag{7b}$$

B. Solution of the partial differential equation for the generating function

The problem of the diffraction of light by a pulsed ultrasonic wave is now reduced to the integration of the second-order partial differential equation (7a) with boundary condition (7b). Once the function $G(\zeta,\eta)$ is known, the amplitudes of the diffracted light waves may be found by expanding $G(\zeta,\eta)$ into a Laurent series or by calculating the integral (6). In order to integrate (7) we propose a solution in the form of a series in ρ_p:

$$G(\zeta,\eta)=\sum_{k=0}^{+\infty}(i\rho_p)^k G_k(\zeta,\eta). \tag{8}$$

Substituting this expansion into (7) and comparing the coefficients of $(i\rho_p)^n$ on both sides, we find the following system of partial differential equations:

$$2\frac{\partial G_0}{\partial\zeta}(\zeta,\eta)-\sum_{j=1}^{+\infty}\alpha_j[\eta^j\exp(-i\delta_j) - \eta^{-j}\exp(i\delta_j)]G_0(\zeta,\eta)=0, \tag{9a}$$

$$2\frac{\partial G_n}{\partial\zeta}(\zeta,\eta) - \sum_{j=1}^{+\infty}\alpha_j[\eta^j\exp(-i\delta_j)-\eta^{-j}\exp(i\delta_j)]G_n(\zeta,\eta) = \eta^2\frac{\partial^2 G_{n-1}}{\partial\eta^2}(\zeta,\eta)+\eta\frac{\partial G_{n-1}}{\partial\eta}(\zeta,\eta),\quad n\geqslant 1, \tag{9b}$$

with boundary conditions

$$G_n(\zeta=0,\eta)=\delta_{n,0}. \tag{9c}$$

The first term of the series expansion (8) is obtained by integration of (9a). The solution satisfying the boundary condition is

$$G_0(\zeta,\eta) = \exp\left(\frac{1}{2}\sum_{j=1}^{\infty}\alpha_j[\eta^j\exp(-i\delta_j)-\eta^{-j}\exp(i\delta_j)]\cdot\zeta\right)$$

or

$$G_0(\zeta,\eta) = \prod_{j=1}^{\infty}\left(\sum_{q_j=-\infty}^{+\infty}J_{q_j}(\alpha_j\cdot\zeta)\eta^{j\cdot q_j}\exp(-iq_j\delta_j)\right), \tag{10}$$

where $J_p(x)$ is the Bessel function of order p.

The second term of (8), $G_1(\zeta,\eta)$, can be calculated from (9) by putting $n=1$. Introducing the notation $t_n=\eta^n\exp(-i\delta_n)$, this term is the solution of the following partial differential equation:

$$2\frac{\partial G_1}{\partial\zeta}(\zeta,\eta)-\sum_{j=1}^{+\infty}\alpha_j(t_j-t_j^{-1})G_1(\zeta,\eta) = G_0(\zeta,\eta)\left(\frac{\zeta}{2}\sum_{j=1}^{\infty}j^2\cdot\alpha_j(t_j-t_j^{-1}) + \frac{\zeta^2}{4}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}j\cdot k\cdot\alpha_j\cdot\alpha_k(t_j+t_j^{-1})(t_k+t_k^{-1})\right).$$

Knowing that $G_1(\zeta=0,\eta)=0$, the second term of the series expansion is given by

$$G_1(\zeta,\eta) = G_0(\zeta,\eta)\left(\frac{\zeta^2}{8}\sum_{j=1}^{\infty}j^2\cdot\alpha_j(t_j-t_j^{-1}) + \frac{\zeta^3}{24}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}j\cdot k\cdot\alpha_j\cdot\alpha_k(t_j+t_j^{-1})(t_k+t_k^{-1})\right). \tag{11}$$

In the same way, we can derive an expression for the third term of the series expansion and after some calculations we find

$$\begin{aligned}G_2(\zeta,\eta) = G_0(\zeta,\eta)\Bigg(&\frac{1}{48}\zeta^3\sum_{j=1}^{+\infty}j^4\cdot\alpha_j(t_j-t_j^{-1}) + \frac{10}{384}\zeta^4\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}j^3\cdot k\cdot\alpha_j\cdot\alpha_k(t_j+t_j^{-1})(t_k+t_k^{-1})\\ &+\frac{7}{384}\zeta^4\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}j^2\cdot k^2\cdot\alpha_j\cdot\alpha_k(t_j-t_j^{-1})(t_k-t_k^{-1})\\ &+\frac{13}{960}\zeta^5\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}\sum_{l=1}^{+\infty}j\cdot k\cdot l^2\cdot\alpha_j\cdot\alpha_k\cdot\alpha_l(t_j+t_j^{-1})(t_k+t_k^{-1})(t_l-t_l^{-1})\\ &+\frac{1}{1152}\zeta^6\sum_{j=1}^{+\infty}\sum_{k=1}^{+\infty}\sum_{l=1}^{+\infty}\sum_{m=1}^{+\infty}j\cdot k\cdot l\cdot m\cdot\alpha_j\cdot\alpha_k\cdot\alpha_l\cdot\alpha_m(t_j+t_j^{-1})(t_k+t_k^{-1})(t_l+t_l^{-1})(t_m+t_m^{-1})\Bigg).\end{aligned} \tag{12}$$

Neglecting terms in ρ_p of higher order than the second one, we have obtained an approximate solution for $G(\zeta,\eta)$:

$$G(\zeta,\eta) = G_0(\zeta,\eta) + i\rho_p \cdot G_1(\zeta,\eta) - \rho_p^2 \cdot G_2(\zeta,\eta). \tag{13}$$

C. Amplitudes of the diffracted light waves

Substituting the expressions (10)–(12) for G_0, G_1, and G_2 into (13) and calculating the coefficient of η^n a second-order approximation formula for the amplitude of the diffracted light wave of order n has been acquired:

$$\Psi_n(\zeta) = \Psi_n^{(0)}(\zeta) + i\rho_p \cdot \Psi_n^{(1)}(\zeta) - \rho_p^2 \cdot \Psi_n^{(2)}(\zeta). \tag{14}$$

A detailed expression for Ψ_n evaluated in the Raman–Nath parameter v is given in the Appendix. The intensities can be calculated from (4).

D. Remarks

(a) $\Psi_n^{(0)}$ given by (A2), corresponds to the amplitude of the nth diffraction order in the case of extreme Raman–Nath regime ($\rho_p = 0$). It is exactly the same expression as Neighbors and Mayer[14] found by means of the diffraction integral theory. As $\Psi_n^{(0)}$ is only the first term in our expression for the diffracted amplitudes, we can consider $i\rho_p \cdot \Psi_n^{(1)}$ and $-\rho_p^2 \cdot \Psi_n^{(2)}$ as corrections to the theory of Neighbors and Mayer.

(b) As an ultrasonic pulse with repetition frequency ν_p^* and center modulation frequency $\nu_0^* = k\nu_p^*$, converges to the continuous plane wave having frequency ν_0^*, the amplitude spectra of the supersonic wave approaches $\mu_k = \mu\alpha_k = \mu$ and $\mu_i = \mu\alpha_i = 0$ for $i \neq k$. The amplitudes of the diffraction pattern in this limit case can be calculated using (14) and are given by

$$\Psi_n(\zeta) = 0, \quad n \notin k\mathbb{Z}$$

$$\begin{aligned}
\Psi_{mk}(\zeta) = J_m(\zeta) + \frac{i\rho_0}{8} \cdot \Big[& \zeta^2 [J_{m-1}(\zeta) - J_{m+1}(\zeta)] + \frac{\zeta^3}{3} [J_{m-2}(\zeta) + 2J_m(\zeta) + J_{m+2}(\zeta)] \Big] \\
- \frac{\rho_0^2}{16} \cdot \Big[& \frac{\zeta^3}{3} (J_{m-1}(\zeta) - J_{m+1}(\zeta)) + \frac{\zeta^4}{4} \left(\frac{17}{6} J_{m-2}(\zeta) + J_m(\zeta) + \frac{17}{6} J_{m+2}(\zeta) \right) \\
& + \frac{13}{60} \zeta^5 [J_{m-3}(\zeta) + J_{m-1}(\zeta) - J_{m+1}(\zeta) - J_{m+3}(\zeta)] \\
& + \frac{1}{72} \zeta^6 [J_{m-4}(\zeta) + 4J_{m-2}(\zeta) + 6J_m(\zeta) + J_{m+2}(\zeta) + J_{m+4}(\zeta)] \Big],
\end{aligned} \tag{15}$$

with $m \in \mathbb{Z}$ and

$$\zeta = \frac{2\pi}{\lambda} \mu z, \quad \rho_0 = \frac{\lambda^2}{\mu_0 \mu \lambda_0^{*2}} = k^2 \cdot \rho_p \quad \left(\lambda_0^* = \frac{\lambda_p^*}{k} \right).$$

This is exactly the same formula, obtained by Kuliasko *et al.*[17] in the case of a single fundamental tone of frequency ν_0^*. As a result of the convergence to a continuous wave, the satellite lines vanish and the diffraction pattern becomes symmetric with respect to the zeroth order.

E. Examples

We considered for all figures in this article two kinds of pulses represented by their functional form $\exp D(x,t;k_1,k_2)$ and Gaus $(x,t;k_1,k_2,k_3)$.

The exponentially damped pulse is characterized by two parameters and causes variations of the refractive index given by formula (1) in which α_j and δ_j satisfy

$$\sum_{j=0}^{N} \alpha_j \sin\left[2\pi j \nu_p^* \left(t - \frac{x}{v_s} \right) + \delta_j \right] = \exp D\left(t - \frac{x}{v_s}; k_1, k_2 \right),$$

with v_s being the velocity of the sound in the medium:

$$\exp D(y;k_1,k_2) = \frac{e^{-\Lambda y}}{N_E(k_1,k_2)} \sin(\omega_0^* y), \quad 0 \leqslant y < a_p,$$

$$\exp D(y;k_1,k_2) = \exp D(y - a_p;k_1,k_2), \quad y \geqslant a_p,$$

where k_1 and k_2 are the repetition and decay parameter defined by

$$a_p = \frac{k_1}{\nu_0^*} = \frac{1}{\nu_p^*} \text{ and } \Lambda^{-1} = \frac{k_2}{\nu_0^*}$$

and $N_E(k_1,k_2)$ is a normalization constant.

The Gaussian-shaped pulse is characterized by three parameters, which are defined as follows:

$$\text{Gaus}(y;k_1,k_2,k_3) = \frac{e^{-1/2[(y-y_2)/y_3]^2}}{N_G(k_1,k_2,k_3)} \sin(\omega_0^* y), \quad 0 \leqslant y < a_p,$$

$$\text{Gaus}(y;k_1,k_2,k_3) = \text{Gaus}(y - a_p;k_1,k_2,k_3), \quad y \geqslant a_p,$$

where

$$a_p = \frac{k_1}{\nu_0^*} = \frac{1}{\nu_p^*}, \quad y_2 = \frac{k_2}{\nu_0^*}, \quad y_3 = \frac{k_3}{\nu_0^*},$$

and $N_G(k_1,k_2,k_3)$ is a normalization constant.

By varying these parameters k_1, k_2 (and k_3), a wide range of pulses can be reached. Nevertheless, the theory is valid for any arbitrary functional form.

In all cases we determined the value for N (the number of Fourier coefficients taken into account) so that

$$\exists j \quad 1 \leqslant j \leqslant N : \alpha_{N+1} < 10^{-5}\alpha_j,$$

with a maximum value of 62.

In order to restrict computer calculations, we suppose that in (14) all Bessel function products $J_{a_1} \cdot J_{a_2} \cdots J_{a_N}$ with

$$\sum_{j=1}^{N} |a_j| > 4$$

vanish (4th order approximation). This means that the Raman–Nath parameters $v_j = \alpha_j v$ are small enough so that terms like $J_1(v_i) \cdot J_2(v_j) \cdot J_2(v_k)$ or $J_{-3}(v_i) \cdot J_2(v_j)$, etc., become negligible.

In Figs. 2 and 3 some results of the generating function method (GEM) [formulas (14) and (4)] are shown. For $\rho_p = 0.0$ the GFM leads to the same diffraction spectrum as the method of Neighbors and Mayer.[14] The corrections due to the additional terms in ρ_p are rather small, but of the same order as the ones Kuliasko *et al.*[17] obtained based on the same method in the case of a continuous wave. The largest corrections occur at the highest diffraction orders (negative and positive).

In Ref. 9, Leroy and Claeys remarked that the GFM for a continuous wave shows a big discrepancy with experiment and other theories when the Raman–Nath parameter v becomes larger. In order to study the validity of expression (15), the authors verified the necessary condition $\Sigma I_n = 1$ in terms of ρ_0 and v. Applying the same control method, we obtained similar relations for the validity of (14) depending on the shape and length of the pulse. For instance in the case of the Gaussian-shaped pulses with parameters $k_1 = 18$, $k_2 = 9$, $k_3 = 1000$ (continuous wave), and $k_3 = 2$, Fig. 4(a) and (b) shows the extension of the region of validity of previous theories for our first and second correction terms. Convinced by many computer calculations, we can conclude that, in general, formula (14) is valid in a region in the ρ_0-v plane, which is larger than in the continuous wave limit, but which converges uniformly toward this limit case as the variance parameter k_3 (or the decay parameter k_2 for an exponentially damped pulse) tends toward infinity.

Nevertheless, as ρ_p has to remain small, the correction terms in (14) never reach values that can change the diffraction pattern in a considerable way. This leads to the conclusion that, in spite of the larger validity region, the GFM does not show important differences compared with the results of Neighbors and Mayer, and within the validity region of the GFM, the restriction of (14) to $\Psi_n^{(0)}(v)$ will lead to satisfying results.

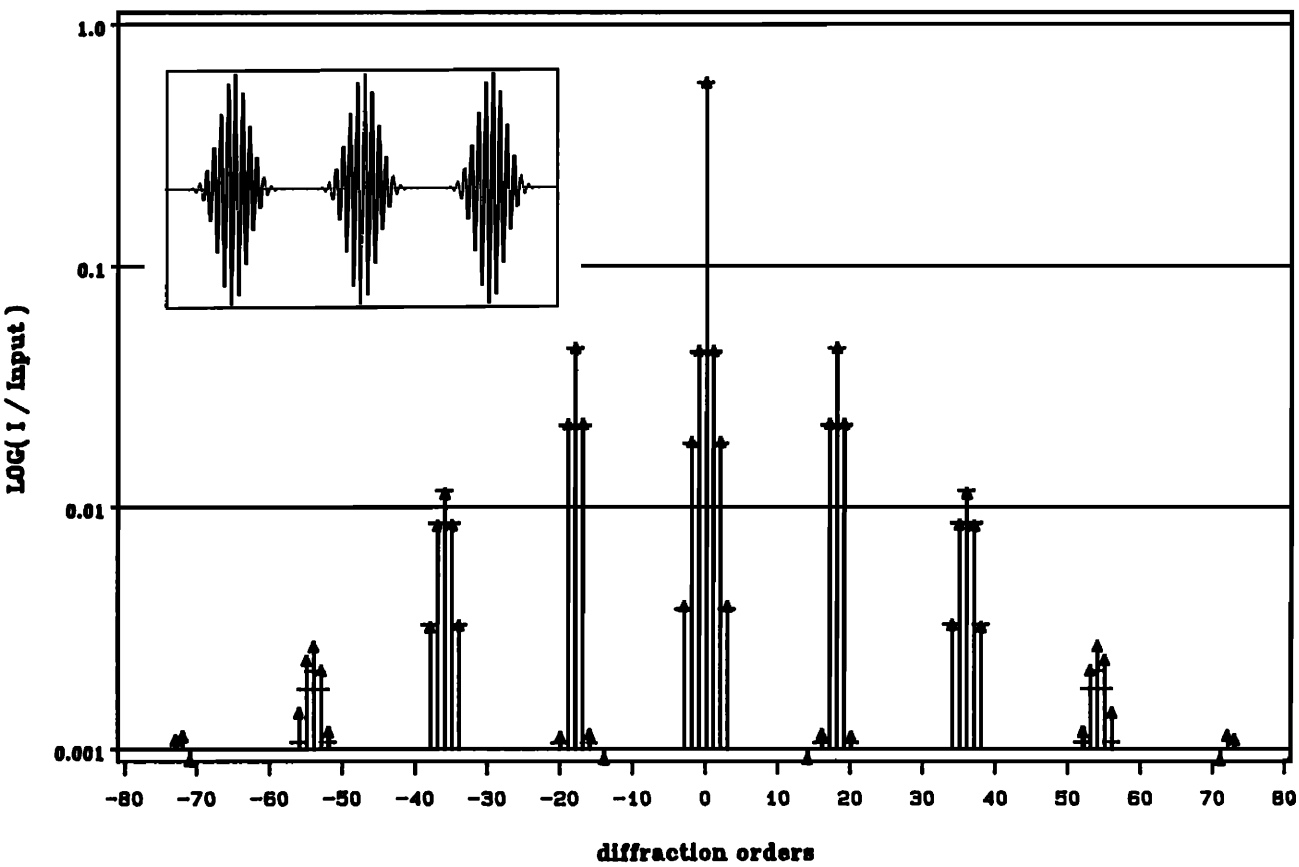

FIG. 2. Far-field diffraction pattern caused by interaction of normally incident light with a Gaussian-shaped ultrasonic pulse ($k_1 = 18$, $k_2 = 9$, $k_3 = 2$) (inset). Results of the generating function method for $\rho_p = 0.0$ (-) and $\rho_p = 9.0\,E$-04 (Δ) ($v = 2.5$).

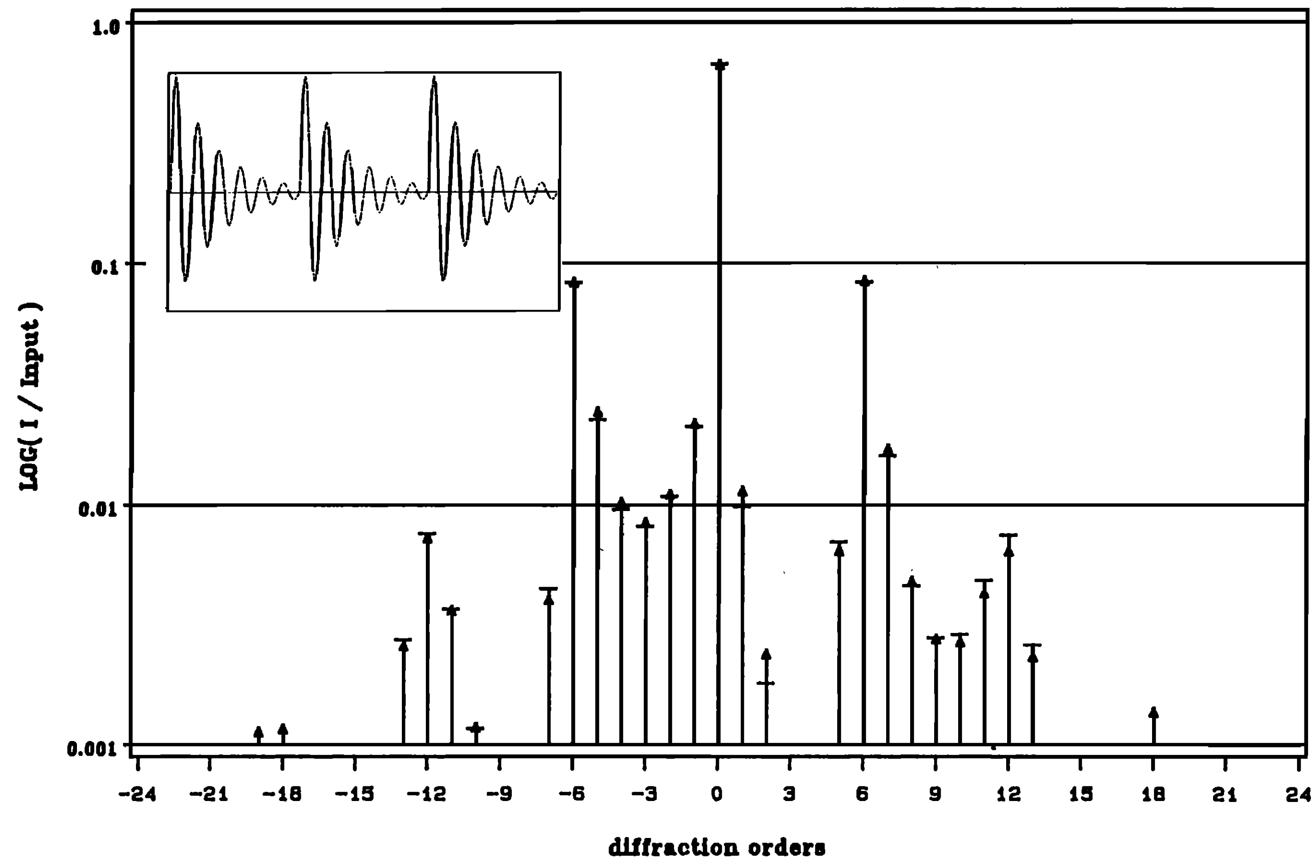

FIG. 3. Far-field diffraction pattern caused by interaction of normally incident light with an exponentially damped ultrasonic pulse ($k_1 = 6$, $k_2 = 2$) (inset). Results of the generating function method for $\rho_p = 0.0$ (-) and $\rho_p = 1.1$ E-02 ($\triangle$) ($v = 2.0$).

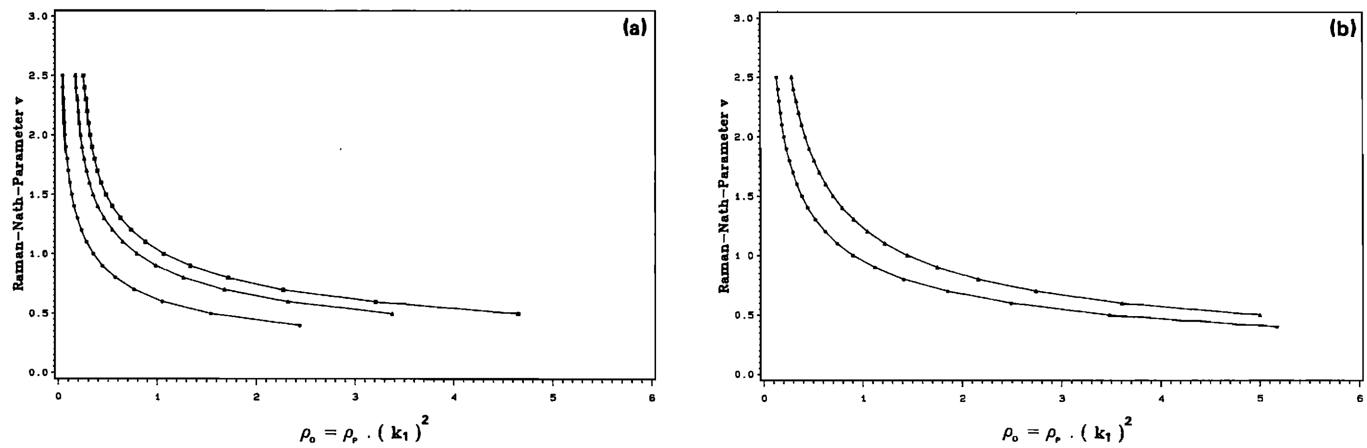

FIG. 4. Limits of validity in terms of ρ_0 ($= k_1^2 \rho_p$) and v for the generating function method calculated by testing the accuracy of the necessary condition $\Sigma_n I_n = 1$: (a) Validity of formula (14) restricted to $\Psi_n^{(0)}$ and first-order correction term $i\rho_p \cdot \Psi_n^{(1)}$, calculated for three Gaussian-shaped pulses ($k_1 = 18, k_2 = 9$) with different variance parameter k_3: upper curve $k_3 = 1$, middle $k_3 = 2$, lowest curve $k_3 = \infty$ ($=$ continuous wave). (b) Validity of formula (14) calculated for two Gaussian-shaped pulses: upper curve $k_1 = 18, k_2 = 9, k_3 = 2$; lower curve $k_1 = 18, k_2 = 9, k_3 = \infty$ ($=$ continuous wave). In both parts the limits of the validity region of the diffraction integral theory of Neighbors and Mayer can be represented by the vertical axis $\rho_0 = 0$.

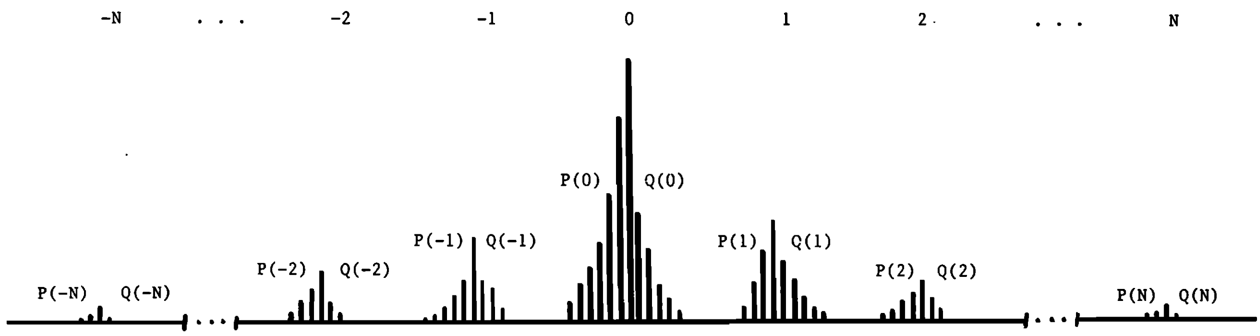

FIG. 5. Illustration of $P(k)$ and $Q(k)$ in a typical far-field diffraction pattern.

III. CALCULATION OF THE FAR-FIELD AMPLITUDES OF THE DIFFRACTION PATTERN BY MEANS OF A LAPLACE TRANSFORM METHOD

A. Transformation of the system of differential equations (2) into an algebraic system

In a previous paragraph we introduced the parameter $k_1 = \nu_0^*/\nu_p^*$. Defining $\beta = \arcsin(\lambda/2\lambda_p^*)$, it is known from theoretical and experimental studies[14,18-20] that the primary orders in the Fraunhöfer diffraction pattern show up in the directions $k_1 \cdot 2\beta$ and at most $k_1 - 1$ secondary orders can be observed between two primary orders at every even multiple of the angle β. Also from previous studies we know that, in the case of normal incidence of the laser beam, as many positive as negative primary diffraction orders will contribute to the final spectrum, while the number of secondary orders left and right of a peak can differ from one primary order compared to another.

Suppose now that only $2N+1$ primary orders (N positive and N negative) have a nonzero contribution of intensity to the final diffraction spectrum and that for the k th primary order ($-N \leqslant k \leqslant N$) only $P(k)$ secondary orders to the left and $Q(k)$ secondary orders to the right of this peak occur (Fig. 5). If we assume that all other amplitudes are vanishing small, the infinite system (2) is reduced to a finite set of equations.

Applying the Laplace transform to the truncated system and substituting

$$\phi_n(s) = \mathscr{L}[\Psi_n(\zeta)] = \int_0^\infty \Psi_n(\zeta) e^{-s\zeta}\, d\zeta, \tag{16}$$

we obtain an algebraic system of equations, which can be written in matrix notation as follows:

$$[2sI + M]\phi(s) = 2E, \tag{17}$$

where I is the unit matrix of dimension d where

$$d = \sum_{k=-N}^{N} [P(k) + Q(k) + 1],$$

$$M = \begin{bmatrix} A(-N) & B(-N,1) & B(-N,2) & & B(-N,2N) \\ -B^*(-N,1) & A(-N+1) & B(-N+1,1) & & \vdots \\ -B^*(-N,2) & -B^*(-N+1,1) & A(-N+2) & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & A(N-1) & B(N-1,1) \\ -B^*(-N,2N) & & & -B^*(N-1,1) & A(N) \end{bmatrix},$$

$$A(k) = \begin{bmatrix} -\rho[kk_1 - P(k)]^2 & z_1 & z_2 & & z_{P(k)+Q(k)} \\ -z_1^* & -\rho[kk_1 - P(k)+1]^2 & z_1 & & \vdots \\ -z_2^* & -z_1^* & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \\ -z_{P(k)+Q(k)}^* & & & -z_1^* & -\rho[kk_1 + Q(k)]^2 \end{bmatrix}, \quad -N \leqslant k \leqslant N,$$

$$B(k,n)=\begin{bmatrix} z_{nk_1+P(k)-P(k+n)} & z_{nk_1+P(k)+Q(k+n)} \\ \vdots & \vdots \\ \vdots & \vdots \\ \vdots & \vdots \\ z_{nk_1-Q(k)-P(k+n)} & z_{nk_1-Q(k)+Q(k+n)} \end{bmatrix}, \quad -N\leqslant k\leqslant N,\ 1\leqslant n\leqslant N-k,$$

with $z_j=\alpha_j \exp(i\delta_j)$;

$$\phi(s)={}^t[\phi_{-Nk_1-P(-N)}(s)\cdots\phi_{-Nk_1+Q(-N)}(s)\phi_{-(N-1)k_1-P(-N+1)}(s)\cdots\phi_{-(N-1)k_1+Q(-N+1)}(s)$$
$$\cdots\phi_{-P(0)}(s)\cdots\phi_0(s)\cdots\phi_{+Q(0)}(s)\cdots\phi_{Nk_1-P(N)}(s)\cdots\phi_{Nk_1+Q(N)}(s)]$$

and

$$E={}^t[0\cdots0\cdots0\cdots0\cdots0\cdots1\cdots0\cdots0\cdots0].$$

B. Solution of the algebraic system with unknown Laplace transformed functions

Defining a new variable $t=-2is$ and multiplying Eq. (17) by i, we get:

$$L(t)\cdot\tilde{\phi}(t)=[D-tI]\tilde{\phi}(t)=2iE, \tag{18}$$

where D is the following matrix:

$$\begin{bmatrix} G(-N) & H(-N,1) & H(-N,2) & & H(-N,2N) \\ H^*(-N,1) & G(-N+1) & H(-N+1,1) & & H(-N+1,2N-1) \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & G(N-1) & H(N-1,1) \\ H^*(-N,2N) & & & H^*(N-1,1) & G(N) \end{bmatrix}$$

and $G(k)=iA(k)$; $H(k,n)=iB(k,n)$; $\tilde{\phi}(t)=\phi(it/2)$.

Also, $G(k)$ is Hermitian and D is also a Hermitian matrix. From linear algebra we know that the zeros $t_1\cdots t_d$ of $\det[L(t)]$ are the eigenvalues of D and, consequently, they are all real. Applying Cramer's method, we can deduce the following expression for the Laplace transformations of the amplitudes:

$$\tilde{\phi}_{kk_1+l_k}(t)=\mathscr{L}[\Psi_{kk_1+l_k}(\zeta)]\left(\frac{it}{2}\right)$$
$$=2i\,\frac{\mathrm{Co}([D-tI]_{P,P_{(k,l_k)}})}{\det[D-tI]},$$
$$-N\leqslant k\leqslant N,\quad -P(k)\leqslant l_k\leqslant Q(k), \tag{19}$$
$$P=\sum_{j=-N}^{-1}[P(j)+Q(j)+1]+P(0)+1,$$
$$P_{(k,l_k)}=\sum_{j=-N}^{k-1}[P(j)+Q(j)+1]+l_k+P(k)+1,$$

and $\mathrm{Co}([D-tI]_{P,P_{(k,l_k)}})$ is the co-factor of the element on the position $(P,P_{(k,l_k)})$ in the matrix $[D-tI]$.

So, generally speaking, numerator and denominator of (19) are polynomials in t with complex coefficients. This means that the Laplace transforms of the amplitudes in the Fraunhöfer region can be expressed in the following way:

$$\phi_{kk_1+l_k}(s)=\mathscr{L}(\Psi_{kk_1+l_k})$$
$$=P_1^{(kk_1+l_k)}(s)/P_2(s), \tag{20}$$

with

$$\deg[P_1^{(kk_1+l_k)}(s)]<\deg[P_2(s)].$$

C. Calculation of the intensities of the diffracted light waves

The inverse Laplace transformation can be calculated applying the Heaviside expansion theorem:[21]

$$\Psi_{kk_1+l_k}(\zeta)=\mathscr{L}^{-1}(\phi_{kk_1+l_k})$$
$$=\sum_{p=1}^{q}\frac{1}{(n_p-1)!}\lim_{s\to s_p}\frac{d^{n_p-1}}{ds^{n_p-1}}$$
$$\times\left(\frac{P_1^{(kk_1+l_k)}(s)}{P_2(s)}(s-s_p)^{n_p}e^{s\zeta}\right),$$

where $s_p=it_p/2$, p: $1\ldots d$, t_p is the eigenvalue of D, q is the number of different eigenvalues (without counting multiplicity), and n_p is the multiplicity of eigenvalue t_p or

$$\Psi_{kk_1+l_k}(\zeta)=\sum_{p=1}^{q}\frac{1}{(n_p-1)!}\lim_{t\to t_p}\frac{d^{n_p-1}}{dt^{n_p-1}}\times\Bigg\{\Bigg[\mathrm{Co}([D-tI]_{P,P_{(k,l_k)}})\times\Bigg(\prod_{\substack{m=1\\ \neq p}}^{q}(t-t_m)^{n_m}\Bigg)^{-1}\Bigg]e^{i/2t\zeta}\Bigg\}. \tag{21}$$

In the case that D has only single eigenvalues, formula (21) can be simplified:

$$\Psi_{kk_1+l_k}(\zeta)=\sum_{p=1}^{d}\mathrm{Co}([D-t_pI]_{P,P_{(k,l_k)}})\times\Bigg(\prod_{\substack{m=1\\ \neq p}}^{d}(t_p-t_m)\Bigg)^{-1}e^{it_p\zeta/2}, \tag{22}$$

and with the following abbreviations:

$$\forall p,\ \forall k,l_k,\ \mathrm{Re}^{(kk_1+l_k)}(t_p)=\mathrm{Re}\big[\mathrm{Co}([D-t_pI]_{P,P_{(k,l_k)}})\big],$$

$$\forall p,\ \forall k,l_k,\ \mathrm{Im}^{(kk_1+l_k)}(t_p)=\mathrm{Im}\big[\mathrm{Co}([D-t_pI]_{P,P_{(k,l_k)}})\big],$$

$$\forall p,\ \mathrm{Pol}(t_p)=\prod_{\substack{m=1\\ \neq p}}^{d}(t_p-t_m).$$

The intensity of order $-(kk_1+l_k)$ can be calculated from the expression (4) and (22) using the double angle formula for cosine and recalling the boundary conditions (2b):

$$\begin{aligned}I_{-(kk_1+l_k)}(\zeta)=\delta_{k,0}\delta_{l_k,0}&-4\sum_{p=1}^{d}\sum_{q>p}^{d}\frac{\mathrm{Re}^{(kk_1+l_k)}(t_p)\mathrm{Re}^{(kk_1+l_k)}(t_q)+\mathrm{Im}^{(kk_1+l_k)}(t_p)\mathrm{Im}^{(kk_1+l_k)}(t_p)}{\mathrm{Pol}(t_p)\mathrm{Pol}(t_q)}\sin^2(t_p-t_q)\frac{\zeta}{4}\\&+2\sum_{p=1}^{d}\sum_{q>p}^{d}\frac{\mathrm{Re}^{(kk_1+l_k)}(t_p)\mathrm{Im}^{(kk_1+l_k)}(t_q)-\mathrm{Re}^{(kk_1+l_k)}(t_q)\mathrm{Im}^{(kk_1+l_k)}(t_p)}{\mathrm{Pol}(t_p)\mathrm{Pol}(t_q)}\sin(t_p-t_q)\frac{\zeta}{2}.\end{aligned} \tag{23}$$

We remark, however, that the most general expression for the intensity is given by the squared modulus of (21), an expression that is rather complicated.

As d generally reaches values higher than 20, the numerical aspect of this method is very important. The use of a subroutine library (NAG) to calculate the eigenvalues of D and the cofactors of $[D-tI]$ makes the method easily programmable.

D. Remarks

In the continuous wave limit numerical calculations have been performed from which we conclude that $I_{-(kk_1+l_k)}(\zeta)$ tends toward zero if $l_k\neq 0$ while $I_{-kk_1}(\zeta)$ is given by exactly the same expressions deduced by Blomme and Leroy,[16] when considering the case of normal incidence. Thus, also by means of this theory, we observe that as a result of the convergence to a continuous wave, the satellite lines vanish and the diffraction pattern becomes symmetric with respect to the zeroth order.

E. Examples

To illustrate the Laplace transform theory we shall consider the same two kinds of pulses described earlier while studying the generating function method. In performing the theoretical calculations N and $\{(P(k),Q(k))|k:-N\cdots N\}$, which determine the nonzero diffraction orders, are chosen so that for any higher-order approximation no significant differences appear. Indeed, using the Laplace transformation method, it is always possible to find a suitable level of approximation to calculate the diffraction pattern very accurately for any kind of frequency range or any kind of shape of the diffracting pulse train, and for no matter what value of the Raman–Nath parameter v. Some results of this powerful numerical model are visualized in Figs. 6 and 7.

Figure 8 illustrates the influence of ρ_p or ρ_0 ($=k_1^2\cdot\rho_p$) on the far-field diffraction spectrum caused by the pulse shown in Fig. 8(a) (reproduced from Wolf *et al.*[20]). Since ρ_p (ρ_0) is proportional to the squared value of the fundamental (center) frequency of the ultrasonic pulse and inverse proportional to the maximum variation μ of the refractive index, any increase of ρ_p (ρ_0) corresponds to an increase of fundamental (center) frequency or a decrease of power of the pulsed ultrasonic wave. The sequence shown in Fig. 8 clearly illustrates the decreasing number of nonzero orders for higher frequencies or lower ultrasonic power. For $\rho_p=1.0\,E\text{-}09$ we nearly get the same results as in the experiment of Wolf *et al.*[20] Up to $\rho_p=1/(k_1)^2\cong 5\,E\text{-}03$ ($\rho_0\approx 1$) the spectrum does not change much. Once $\rho_p>1\,E\text{-}02$ ($\rho_0\approx 2$) the changes become larger and larger: the number of primary orders diminishes, satellite lines vanish, and when ρ_p is large enough only zeroth, first, and minus first primary orders are measurable. An analog change due to ρ_p is illustrated in Fig. 6.

The effect of increasing v, shown by Neighbors and Mayer,[14] can also be found using the Laplace transform the-

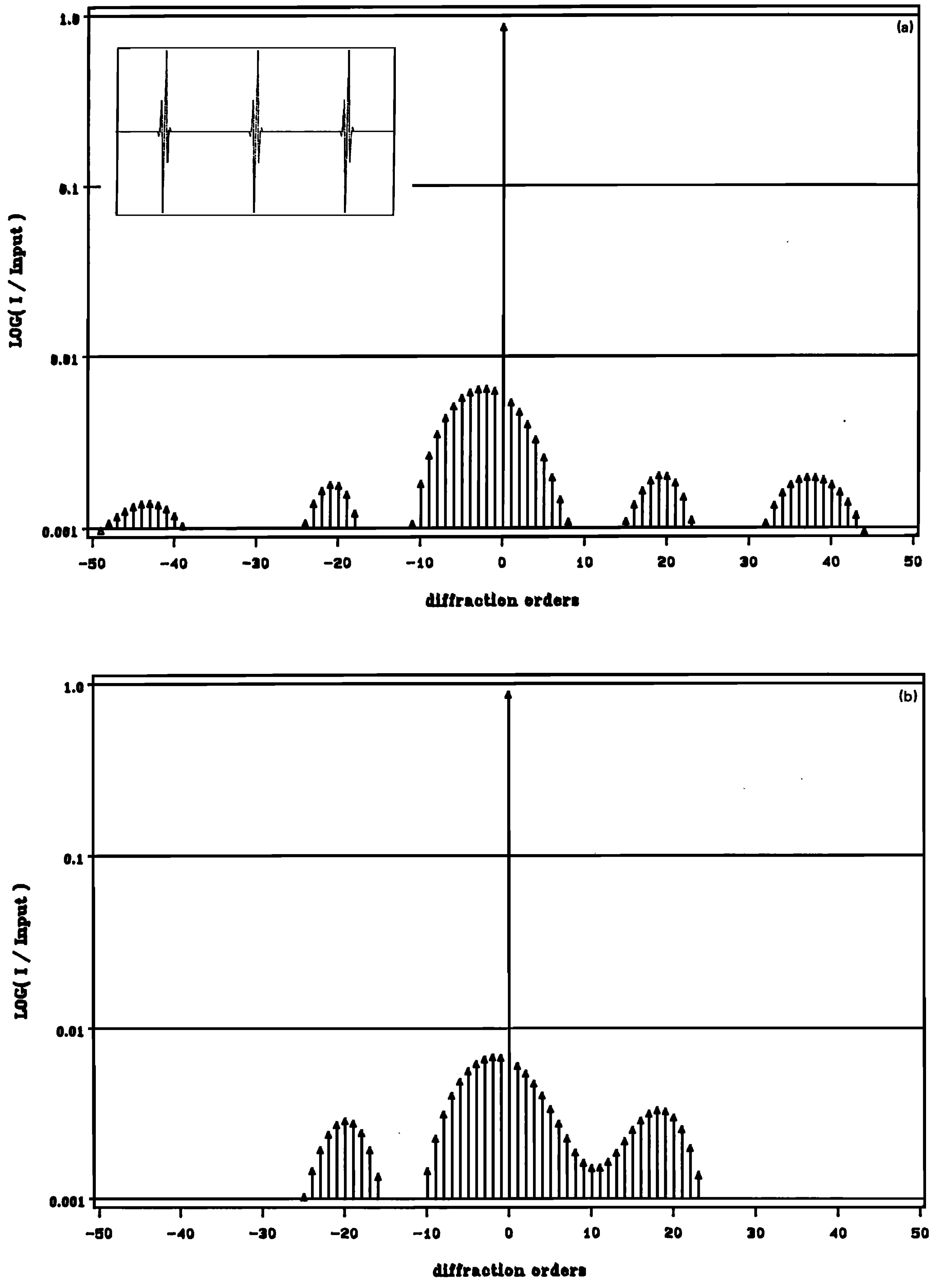

FIG. 6. Far-field diffraction pattern caused by a Gaussian-shaped ultrasonic pulse $k_1 = 20$, $k_2 = 10$, $k_3 = 0.5$ (inset): Results of the Laplace transform theory ($v = 3.0$): (a) $\rho_p = 1.0\,E\text{-}08$ or $\rho_0 = 4.0\,E\text{-}06$; (b) $\rho_p = 2.5\,E\text{-}03$ or $\rho_0 = 1.0$.

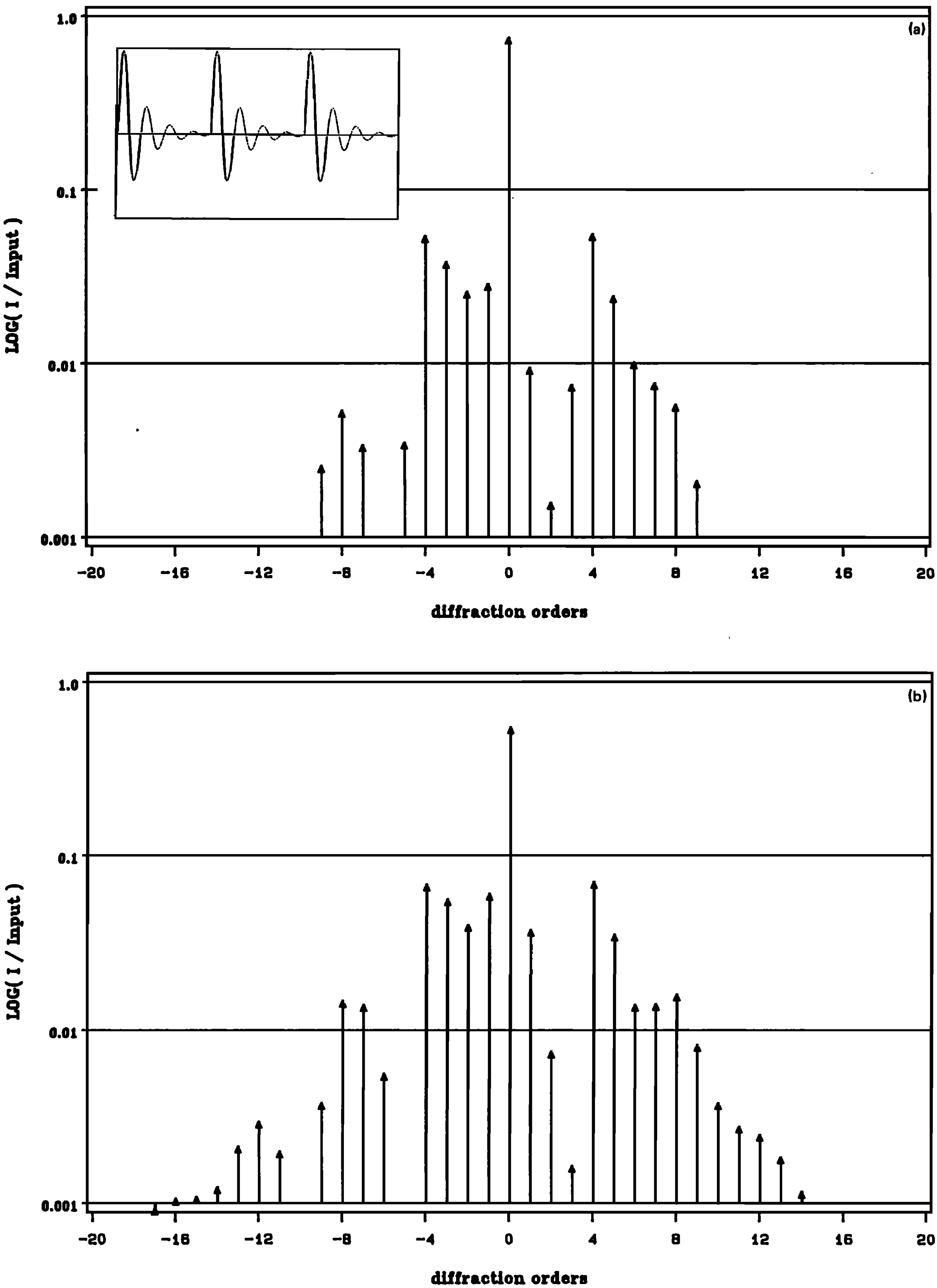

FIG. 7. Far-field diffraction pattern caused by an exponentially damped ultrasonic pulse $k_1 = 4$, $k_2 = 0.9$ (inset): Results of the Laplace transform theory ($\rho_p = 1.0\,E\text{-}09$ or $\rho_0 = 1.6\,E\text{-}08$): (a) $v = 2.0$, (b) $v = 3.0$.

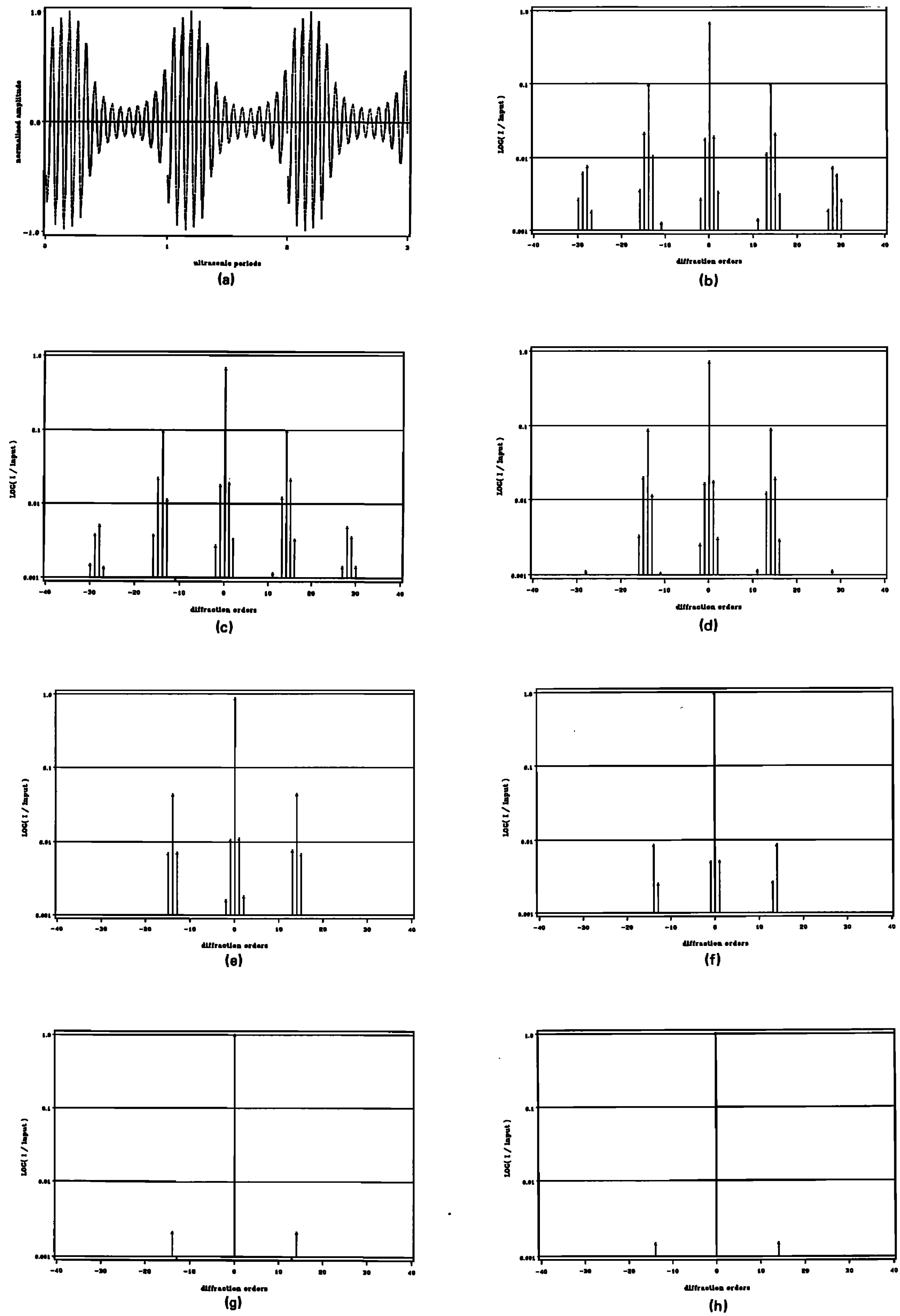

FIG. 8. Dependence of the far-field diffraction pattern on the parameter ρ, in the case of a diffracting pulse shown in (a) (reproduced from Wolf *et al.*[20]): Results of the Laplace transform theory ($v = 1.6$): (b) $\rho_p = 1.0\,E\text{-}07$ or $\rho_0 = 2.19\,E\text{-}05$, (c) $\rho_p = 5.0\,E\text{-}03$ or $\rho_0 = 1.09$, (d) $\rho_p = 1.0\,E\text{-}02$ or $\rho_0 = 2.19$, (e) $\rho_p = 2.0\,E\text{-}02$ or $\rho_0 = 4.38$, (f) $\rho_p = 3.0\,E\text{-}02$ or $\rho_0 = 6.57$, (g) $\rho_p = 7.0\,E\text{-}02$ or $\rho_0 = 15.33$, (h) $\rho_p = 9.0\,E\text{-}02$ or $\rho_0 = 19.71$.

ory. Figures 7 and 9 clearly illustrate that more and more orders show up when v becomes larger.

Choosing N and $\{[P(k), Q(k)] \mid k: -N\cdots N\}$ in an appropriate way, we can calculate the spectrum in the limit $\rho_p = 0.0$ (when most orders appear): Figs. 6(a), 7, and 8(b). Comparing Figs. 7(a) and 8(b) with Fig. 10(a) and (b), which are the results according to Neighbors and Mayer,[14] we may conclude that the correspondence when $\rho_p \rightarrow 0.0$ is extremely good.

At last, we visualize the convergence of our new Laplace transform theory toward the *MN–OA* method (with $M = N$ for normal incidence) of Blomme and Leroy[16] as a pulse converges to a continuous wave (Fig. 11). Increasing the variance parameter k_3 of a Gaussian-shaped pulse, the diffraction pattern changes in this way that satellite lines lose part of their intensity while this loss of energy is compensated by an increase of the intensity of the primary orders (except for the zeroth order). When k_3 is large enough, the far-field spectrum is exactly the same as the one obtained by the *MN–OA* method of Blomme and Leroy for normal incidence of light on a continuous ultrasonic wave: no satellite lines and a perfect symmetric pattern.

IV. GENERAL THEORY CONCERNING SYMMETRY PROPERTIES OF THE DIFFRACTION PATTERN

Theoretical and experimental results[14,18–20] clearly show an asymmetric diffraction pattern when light is normal incident on an ultrasonic pulse. Neighbors and Mayer explained this property by analyzing successive approximations of their expression for the amplitudes of the diffraction orders (which is equal to the first term in the series expansion for the amplitudes obtained by our generating function method). In this paragraph, we derive a general condition concerning the symmetry property of the far-field pattern caused by the diffraction of a normally incident plane laser beam by a superposed ultrasonic wave. The method we use here is based on the generating function method, which was explained earlier.

Considering the amplitudes $\Psi_n(\zeta)$ as the coefficients of the Laurent expansion of the unknown generating function $G(\zeta,\eta)$, we showed that $G(\zeta,\eta)$ satisfies the partial differential equation (7a) with boundary condition (7b).

Replacing the complex variable η by $\exp(iA)\,\eta^{-1}$ (A being any real number) in $G(\zeta,\eta)$ we find that

$$\widetilde{G}(\zeta,\eta) = G[\zeta,\exp(iA)\eta^{-1}] = \sum_{n=-\infty}^{+\infty} \Psi_{-n}(\zeta)\exp(inA)\eta^n,$$

satisfies the following partial differential equation:

$$2\frac{\delta\widetilde{G}}{\delta\zeta} - \sum_{j=1}^{+\infty} \alpha_j\{\eta^j \exp[i\delta_j - ijA + i(2k+1)\pi] - \eta^{-j}\exp[-i\delta_j + ijA + i(2l+1)\pi]\}\widetilde{G} = i\rho_p\left(\eta^2\frac{\delta^2\widetilde{G}}{\delta\eta^2} + \eta\frac{\delta\widetilde{G}}{\delta\eta}\right)$$

(k and l being any integer) with boundary conditions $\widetilde{G}(0,\eta) = 1$. Knowing this, we can derive the necessary and sufficient conditions for the symmetry of the diffraction spectrum:

$$A = 2\delta_1 + (2k-1)\pi, \quad k\in\mathbb{Z} \tag{24a}$$

$$\delta_j = j\delta_1 + [(2l-1) - j](\pi/2), \quad l\in\mathbb{Z}. \tag{24b}$$

Indeed, if these conditions are fulfilled, the functions $G(\zeta,\eta)$ and $\widetilde{G}(\zeta,\eta)$ satisfy the same partial differential equation with the same boundary condition. As the solution is unique, we may conclude that under these restrictions

$$\Psi_n(\zeta) = \Psi_{-n}(\zeta)\exp(inA),$$

which means that the diffraction pattern is symmetric with respect to the zeroth order.

In the case $\delta_1 = 0$ (where δ_j is the phase difference between the jth harmonic and the fundamental tone), we find that

$$\Psi_n = (-1)^n\Psi_{-n},$$

only if

$$\delta_j = [(2l-1) - j](\pi/2) \quad (l\in\mathbb{Z}, j = 2,3,...),$$

a condition that has been obtained by Mertens and Leroy.[22]

The general conditions (24) mean that the diffraction pattern obtained by diffraction of light by ultrasonic pulses at normal incidence is symmetric only if the Fourier phases in the sine expansion of the pulsed wave satisfy the following simple rule: if j is odd $\delta_j - j\delta_1$ must be an even multiple of $\pi/2$ and if j is even $\delta_j - j\delta_1$ must be an odd multiple of $\pi/2$. As these conditions are rather strong, they are generally not fulfilled in the case of pulses, so that mostly the diffraction pattern will be asymmetric.

V. CONCLUSION

The diffraction of light, normally incident on a pulsed ultrasonic wave, has been studied using two generalized approximation methods. Both theories clearly illustrate that compared with the case of a continuous wave, more diffraction orders (satellite lines) appear while the spectrum itself becomes asymmetric.

By means of the generating function method, we obtain an analytical expression for the amplitudes of the diffracted light waves that consists of a series expansion in a parameter ρ_p, with the zeroth-order term being exactly the formula known from earlier work. The corrections found by evaluating this series up to the second order, are rather small and only valuable for less or more strong restrictions.

On the other hand, a much more powerful numerical method based on Laplace transformations was presented. This method can always be adjusted in such a way that it is possible to cover as well the Raman–Nath-like diffraction for low frequencies as the Bragg type interaction for high frequencies. Perhaps the only disadvantage is that we do not have an analytical expression for the amplitudes but a numerical one leading to intensive, but nevertheless easily programmable computer work. Different sets of examples have been provided that demonstrate the influence of the ultra-

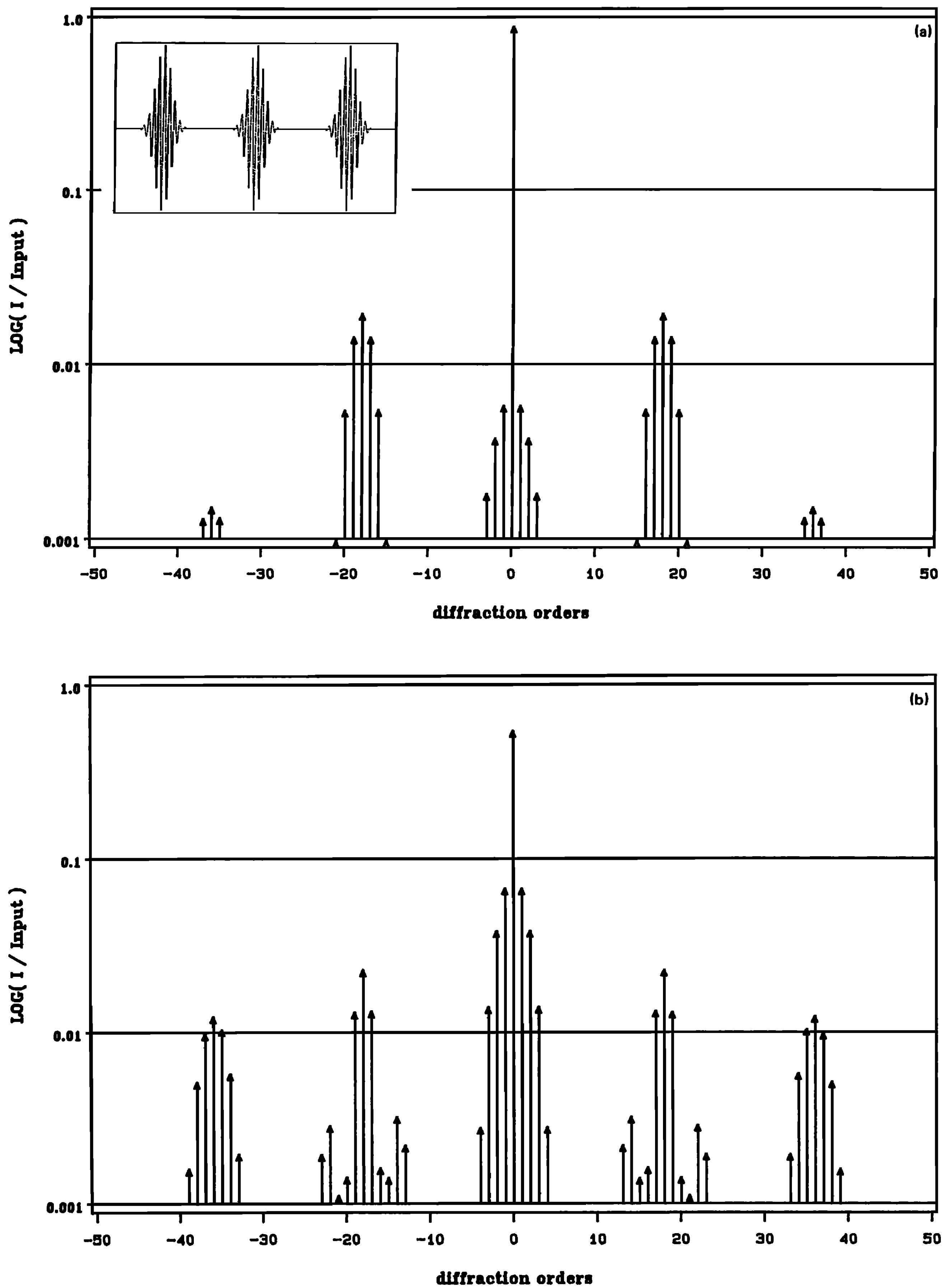

FIG. 9. Dependence of the far-field diffraction pattern on the Raman–Nath parameter v in the case of a Gaussian-shaped pulse $k_1 = 18$, $k_2 = 9$, $k_3 = 1.5$ (inset): Results of the Laplace transform theory ($\rho_p = 1.0\,E\text{-}03$ or $\rho_0 = 0.324$): (a) $v = 1.5$, (b) $v = 3.5$.

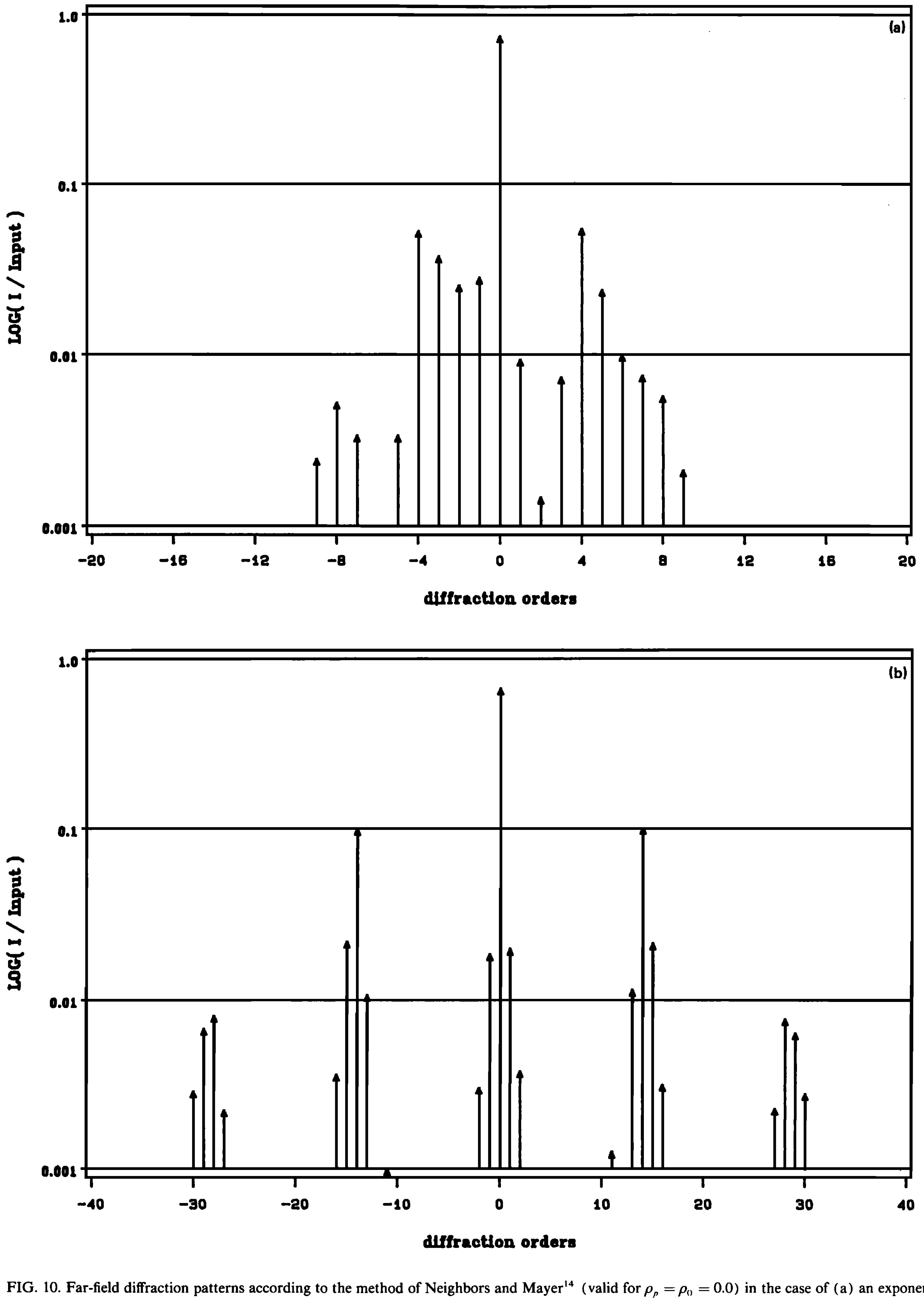

FIG. 10. Far-field diffraction patterns according to the method of Neighbors and Mayer[14] (valid for $\rho_p = \rho_0 = 0.0$) in the case of (a) an exponentially damped pulse $k_1 = 4$, $k_2 = 0.9$ with $v = 2.0$, (b) the pulse of Fig. 8(a) with $v = 1.6$. The corresponding results according to the Laplace transform theory for $\rho_p \to 0$ are visualized in Figs. 7(a) and 8(b).

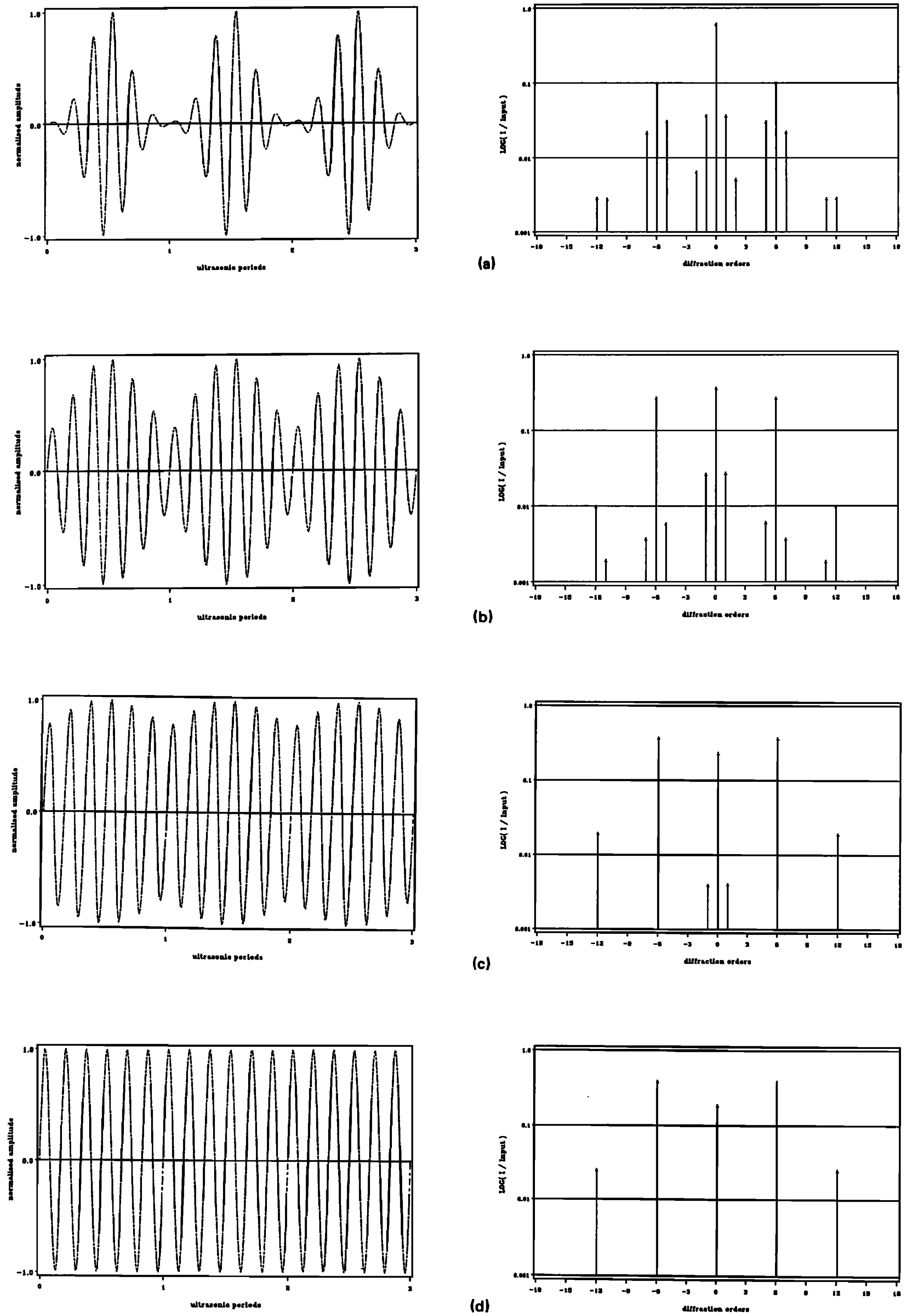

FIG. 11. Convergence of the new Laplace transform method toward the *MN–OA* method.[16] On the left side: Convergence of a Gaussian-shaped pulse ($k_1 = 6$, $k_2 = 3$, $k_3 =$ variable) to a continuous wave: (a) $k_3 = 1$, (b) $k_3 = 2$, (c) $k_3 = 4$, and (d) continuous wave. On the right side: Convergence of the corresponding diffraction patterns to the spectrum of a continuous wave; $v = 2.0$, $\rho_p = 4.166\,E\text{-}02$, or $\rho_0 = 1.5$.

sonic frequency (included in the parameters ρ_p and ρ_0), the Raman–Nath parameter v, and the shape of the pulse (spectral composition).

The general property of asymmetry of the diffraction pattern caused by an arbitrary pulsed ultrasonic wave at normal incidence has been explained theoretically.

ACKNOWLEDGMENTS

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APPENDIX: DETAILED EXPRESSIONS FOR $\Psi_n(v)$ [$v=(2\pi/\lambda)\mu L$]

Define

$$v_j = \alpha_j v, q = \sum_{k=2}^{N} k q_k$$

and N the number of coefficients taken into account in the Fourier expansion of the time history of the pulse:

$$\begin{aligned}
\Psi_n(v) = \sum_{q_2\cdots q_N=-\infty}^{+\infty} & J_{q_2}(v_2)\cdots J_{q_N}(v_N)\exp\left(-i\sum_{k=2}^{N} q_k\delta_k\right)\bigg\{J_{n-q}(v_1)\exp(-i(n-q)\delta_1) \\
&+\sum_{j=1}^{N}\alpha_j\cdot j^2\left(i\rho_p\cdot\frac{v^2}{8}-\rho_p^2\cdot j^2\,\frac{v^3}{48}\right)\{J_{n-q-j}(v_1)\exp[-i\delta_j-i(n-q-j)\delta_1] \\
&-J_{n-q+j}(v_1)\exp[+i\delta_j-i(n-q+j)\delta_1]\} \\
&+\sum_{j=1}^{N}\sum_{k=1}^{N}\alpha_j\cdot\alpha_k\cdot j\cdot k\bigg[\left(i\rho_p\cdot\frac{v^3}{24}-\rho_p^2\cdot j^2\,\frac{10}{384}\,v^4-\rho_p^2\cdot j\cdot k\,\frac{7}{384}\,v^4\right) \\
&\times j_{n-q-j-k}(v_1)\exp[-i\delta_j-i\delta_k-i(n-q-j-k)\delta_1] \\
&\times\left(i\rho_p\cdot\frac{v^3}{24}-\rho_p^2\cdot j^2\,\frac{10}{384}\,v^4+\rho_p^2\cdot j\cdot k\,\frac{7}{384}\,v^4\right)J_{n-q+j-k}(v_1)\exp[i\delta_j-i\delta_k-i(n-q+j-k)\delta_1] \\
&\times\left(i\rho_p\cdot\frac{v^3}{24}-\rho_p^2\cdot j^2\,\frac{10}{384}\,v^4+\rho_p^2\cdot j\cdot k\,\frac{7}{384}\,v^4\right)J_{n-q-j+k}(v_1)\exp[-i\delta_j+i\delta_k-i(n-q-j+k)\delta_1] \\
&\times\left(i\rho_p\cdot\frac{v^3}{24}-\rho_p^2\cdot j^2\,\frac{10}{384}\,v^4-\rho_p^2\cdot j\cdot k\,\frac{7}{384}\,v^4\right)J_{n-q+j+k}(v_1)\exp[i\delta_j+i\delta_k-i(n-q+j+k)\delta_1]\bigg] \\
&+\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\alpha_j\cdot\alpha_k\cdot\alpha_1\cdot j\cdot k\cdot l^2\left(-\rho_p^2\cdot v^5\cdot\frac{13}{960}\right) \\
&\times\{J_{n-q-j-k-l}(v_1)\exp[-i\delta_j-i\delta_k-i\delta_l-i(n-q-j-k-l)\delta_1] \\
&+J_{n-q+j-k-l}(v_1)\exp[+i\delta_j-i\delta_k-i\delta_l-i(n-q+j-k-l)\delta_1] \\
&+J_{n-q-j+k-l}(v_1)\exp[-i\delta_j+i\delta_k-i\delta_l-i(n-q-j+k-l)\delta_1] \\
&+J_{n-q+j+k-l}(v_1)\exp[+i\delta_j+i\delta_k-i\delta_l-i(n-q+j+k-l)\delta_1] \\
&-J_{n-q-j-k+l}(v_1)\exp[-i\delta_j-i\delta_k+i\delta_l-i(n-q-j-k+l)\delta_1] \\
&-J_{n-q+j-k+l}(v_1)\exp[+i\delta_j-i\delta_k+i\delta_l-i(n-q+j-k+l)\delta_1] \\
&-J_{n-q-j+k+l}(v_1)\exp[-i\delta_j+i\delta_k+i\delta_l-i(n-q-j+k+l)\delta_1] \\
&-J_{n-q+j+k+l}(v_1)\exp[+i\delta_j+i\delta_k+i\delta_l-i(n-q+j+k+l)\delta_1]\} \\
&+\sum_{j=1}^{N}\sum_{k=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N}\alpha_j\cdot\alpha_k\cdot\alpha_l\cdot\alpha_m\cdot j\cdot k\cdot l\cdot m\cdot\left(-\rho_p^2\cdot v^6\cdot\frac{1}{1152}\right) \\
&\times\Big[J_{n-q-j-k-l-m}(v_1)\exp[-i\delta_j-i\delta_k-i\delta_l-i\delta_m-i(n-q-j-k-l-m)\delta_1] \\
&+J_{n-q+j-k-l-m}(v_1)\exp[i\delta_j-i\delta_k-i\delta_l-i\delta_m-i(n-q+j-k-l-m)\delta_1] \\
&+J_{n-q-j+k-l-m}(v_1)\exp[-i\delta_j+i\delta_k-i\delta_l-i\delta_m-i(n-q-j+k-l-m)\delta_1] \\
&+J_{n-q+j+k-l-m}(v_1)\exp[i\delta_j+i\delta_k-i\delta_l-i\delta_m-i(n-q+j+k-l-m)\delta_1] \\
&+J_{n-q-j-k+l-m}(v_1)\exp[-i\delta_j-i\delta_k+i\delta_l-i\delta_m-i(n-q-j-k+l-m)\delta_1]
\end{aligned}$$

$$+J_{n-q+j-k+l-m}(v_1)\exp[i\delta_j - i\delta_k + i\delta_l - i\delta_m - i(n-q+j-k+l-m)\delta_1]$$
$$+J_{n-q-j+k+l-m}(v_1)\exp[-i\delta_j + i\delta_k + i\delta_l - i\delta_m - i(n-q-j+k+l-m)\delta_1]$$
$$+J_{n-q+j+k+l-m}(v_1)\exp[i\delta_j + i\delta_k + i\delta_l - i\delta_m - i(n-q+j+k+l-m)\delta_1]$$
$$+J_{n-q-j-k-l+m}(v_1)\exp[-i\delta_j - i\delta_k - i\delta_l + i\delta_m - i(n-q-j-k-l+m)\delta_1]$$
$$+J_{n-q+j-k-l+m}(v_1)\exp[i\delta_j - i\delta_k - i\delta_l + i\delta_m - i(n-q+j-k-l+m)\delta_1]$$
$$+J_{n-q-j+k-l+m}(v_1)\exp[-i\delta_j + i\delta_k - i\delta_l + i\delta_m - i(n-q-j+k-l+m)\delta_1]$$
$$+J_{n-q+j+k-l+m}(v_1)\exp[i\delta_j + i\delta_k - i\delta_l + i\delta_m - i(n-q+j+k-l+m)\delta_1]$$
$$+J_{n-q-j-k+l+m}(v_1)\exp[-i\delta_j - i\delta_k + i\delta_l + i\delta_m - i(n-q-j-k+l+m)\delta_1]$$
$$+J_{n-q+j-k+l+m}(v_1)\exp[+i\delta_j - i\delta_k + i\delta_l + i\delta_m - i(n-q+j-k+l+m)\delta_1]$$
$$+J_{n-q-j+k+l+m}(v_1)\exp[-i\delta_j + i\delta_k + i\delta_l + i\delta_m - i(n-q-j+k+l+m)\delta_1]$$
$$+J_{n-q+j+k+l+m}(v_1)\exp[+i\delta_j + i\delta_k + i\delta_l + i\delta_m - i(n-q+j+k+l+m)\delta_1]\Big]\Big\}. \qquad \text{(A1)}$$

Expressions for $\Psi_n^{(1)}(v)$ and $\Psi_n^{(2)}(v)$ can be deduced from (A1) and (14) by comparing terms in $i\rho_p$ and $-\rho_p^2$. We only explicitly write the formula for $\Psi_n^{(0)}(v)$:

$$\Psi_n^{(0)}(\zeta) = \sum_{q_2=-\infty}^{+\infty} \sum_{q_3=-\infty}^{+\infty} \cdots \sum_{q_N=-\infty}^{+\infty} J_{q_2}(v_2) \times J_{q_3}(v_3)\cdots J_{q_N}(v_N) J_{n-q}(v_1) \times \exp\left(-i\sum_{k=2}^{N} q_k\delta_k - i(n-q)\delta_1\right). \qquad \text{(A2)}$$

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